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Green and Gotzmann theorems for polynomial rings with restricted powers of the variables

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Abstract

We study the Hilbert functions of strongly stable ideals in polynomial rings with restricted powers of the variables. © 1998 Elsevier Science B.V. All rights reserved.

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1. Introduction

The extremal properties of Hilbert functions have been studied extensively. One of the main reasons for the fertility and appeal of this subject is that one can study Hilbert functions using methods and techniques from several mathematical areas: combinatorics, commutative algebra, and algebraic geometry. In [13] Macaulay characterized the Hilbert functions of quotients of polynomial rings, or equivalently, the h-vectors of multicomplexes [14, Section 2.2]. Given Macaulay's result, it is natural to ask whether vector spaces of forms of the same degree which achieve Macaulay's bound enjoy some other special properties. In [7] Gotzmann proved his remarkable Persistence Theorem which states that such extremal vector spaces in degree d generate extremal vector spaces in degree $d+1$. We will call such vector spaces *Gotzmann*. Structure results about Gotzmann vector spaces have been obtained in [3, 6, 8]. Green [8] characterized the Hilbert functions of rings obtained by moding out quotients of polynomial rings with fixed Hilbert function by a general linear form. A result of Kruskal [12] and Katona [11] extended the study of the extremal properties of Hilbert functions to rings other than the polynomial rings. They characterized the f-vectors of simplicial

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complexes, or equivalently, the Hilbert functions of quotients of rings of the form $k[x_1,\dots,x_n]/(x_1^2,\dots,x_n^2)$. Since it does not make any difference if the variables commute or anticommute, this also characterizes the Hilbert functions of quotients of exterior algebras (see also [1]). Clements and Lindström [4] generalized both Macaulay's and Kruskal–Katona's results to rings of the form $R=k[x_1,\dots,x_n]/(x_1^{a_1},\dots,x_n^{a_n})$, where k is a field, $2\le a_1\le a_2\le\cdots\le a_n\le\infty$, and $x_i^\infty=0$. We will extend the definition of a Gotzmann vector space to extremal vector spaces in any such ring R. A vector space $V\subseteq R$ (resp. an ideal $I\subseteq R$) is called *strongly stable*, if V (resp. I) is generated by monomials and whenever $x_im\in V$ (resp. $x_im\in I$) for some monomial m, then $x_jm\in V$ (resp. $x_jm\in I$) for any $j\le i$. In her dissertation 1995 Bigatti gave a new proof of Gotzmann Persistence Theorem for polynomial rings in characteristic 0. She proved the theorem for strongly stable vector spaces and used Gröbner basis theory to reduce the general case to that of strongly stable vector spaces. Aramova et al. [1] showed that with minor modifications Gröbner basis theory known from polynomial rings carries over to exterior algebras. They used an approach similar to Bigatti's to prove a Persistence Theorem for Gotzmann vector spaces in exterior algebras.

It is not hard to see that to prove Macaulay's, Green's, and Kruskal–Katona's theorems it is enough to consider strongly stable vector spaces. Moreover, in the sense of Green's theorem, the last variable x_n is a general linear form for any strongly stable vector space.

In this paper we generalize Green's theorem (in Theorem 2.1(1)), Clements–Lindström theorem (in Theorem 2.1(2)), and Gotzmann and Aramova–Herzog–Hibi Persistence theorems (in Theorem 2.1(3)) to strongly stable ideals in rings of the form

$$k[x_1,\dots,x_n]/(x_1^{a_1},\dots,x_n^{a_n}), \tag{1}$$

where $2\le a_i\le\infty$ for $1\le i\le n$. (We are not assuming that $a_1\le a_2\le\cdots\le a_n$.)

Unlike Gotzmann and Aramova–Herzog–Hibi Persistence theorems, it is necessary to assume in Theorem 2.1(3) that $\dim VS_i=\dim LS_i$ for more values of i than just $i=1$ as the following example shows:

Example 1. Let $S=k[x_1,x_2,x_3]/(x_1^3,x_2^3,x_3^3)$, V be the strongly stable vector space spanned by x_1^2,x_1x_2,x_2^2, and L the lexicographic vector space spanned by x_1^2,x_1x_2,x_1x_3. Then $\dim V=\dim L=3$, $\dim VS_1=\dim LS_1=5$, but $\dim VS_2=6>\dim LS_2=5$.

Specializing our proofs to the case of polynomial rings ($a_1=a_2=\cdots=a_n=\infty$) one obtains new proofs of Macaulay's and Green's theorems. Since our proofs work for anticommuting, as well as for commuting indeterminates, we can also specialize to the case of exterior algebras ($a_1=a_2=\cdots=a_n=2$; anticommuting indeterminates) and obtain a new proof of Aramova–Herzog–Hibi Persistence Theorem.

2. Hilbert functions of strongly stable ideals

Let S be a ring of the form (1). We denote by S_d the vector space of homogeneous polynomials of degree d in S. Let $\tilde{S}=k[x_1,\dots,x_{n-1}]/(x_1^{a_1},\dots,x_{n-1}^{a_{n-1}})\subset S$ and let $V\subseteq S_d$ be a vector space generated by monomials. Let $\tilde{V}$ be the vector space generated by the monomials in V which are not divisible by x_n and $V'=\{f/x_n\colon f\in V \text{ and } x_n\mid f\}$, so $V=\tilde{V}\oplus x_nV'$. If, in addition, V is strongly stable, then $x_iV'\subseteq V$ for any i, so $V'S_1\subseteq V$. Then $VS_1=\tilde{V}\tilde{S}_1+\tilde{V}x_n+x_nV'S_1\subseteq\tilde{V}\tilde{S}_1+\tilde{V}x_n+x_nV=\tilde{V}\tilde{S}_1\oplus x_nV\subseteq VS_1$, so $VS_1=\tilde{V}\tilde{S}_1\oplus x_nV$. The main result in this paper is:

Theorem 2.1. *Let V, $L\subseteq S_d$ be vector spaces such that V is strongly stable, L is generated by an initial lex-segment, and* $\dim V=\dim L$. *Then*:

(1) $\dim\tilde{V}\geq\dim\tilde{L}$;
(2) $\dim VS_1\geq\dim LS_1$;
(3) *Let* $u=\max\{1,a_i-1\mid 1\leq i\leq n,\ a_i<\infty\}$. *If* $\dim VS_i=\dim LS_i$ *for* $1\leq i\leq u$, *then* $\dim VS_j=\dim LS_j$ *for all* $j\geq 1$.

In the proof of this theorem, we use the following Theorem 2.2 about multicomplexes with restricted multiplicities.

If $C\subseteq S_d$ is a set of monomials and $m\in S$ is a monomial, we set $mC=\{mm'\colon m'\in C\}$ and $\phi(m)=\max\{i\colon x_1^i\mid m\}$. We also denote by $C^{(i)}$, $0\leq i\leq a_1-1$, the set $C^{(i)}=\{m/x_1^i\colon m\in C \text{ and } \phi(m)=i\}$. We set $C'=\bigcup_{i=1}^{a_1-1}x_1^iC^{(i)}=\{m\in C\colon x_1\mid m\}$ and $\Delta C=\{m\in S_{d-1}\colon m$ divides a monomial in $C\}$. (So $\Delta C=\emptyset$ when $d=0$.) Then $C=\bigcup_{i=0}^{a_1-1}x_1^iC^{(i)}=C^{(0)}\cup C'$.

Theorem 2.2. *Let $C,R\subseteq S_d$ be sets of monomials such that C is strongly stable and R is an initial rev-lex segment with $|R|=|C|$. Then*

(1) $|C^{(0)}|\leq|R^{(0)}|$;
(2) $|\Delta C|\geq|\Delta R|$;
(3) *If* $|\Delta C|=|\Delta R|$, *then* $|C^{(0)}|=|R^{(0)}|$.

3. Proofs

To prove Theorems 2.1 and 2.2 we will need two preliminary lemmas about the rev-lex order.

Lemma 3.1. *If $m_1>m_2$ are two consecutive (with respect to the rev-lex order) monomials in S_d, then either $\phi(m_2)=\phi(m_1)-1$, or $\phi(m_2)\geq\phi(m_1)$.*

Proof. Let $m_1=x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n}$, so $\phi(m_1)=i_1$. Since m_1 is not the least monomial in S_d, it follows that there exists some $j\geq 2$, such that $i_j<a_j-1$. Let u be the least such j. If $u=2$, then $m_2=x_1^{i_1-1}x_2^{i_2+1}x_3^{i_3}\cdots x_n^{i_n}$, so $\phi(m_1)=\phi(m_2)+1$.

If $u>2$, then $m_2=x_1^{r_1}x_2^{r_2}\cdots x_n^{r_n}$, where $r_u=i_u+1$, $r_j=i_j$ for $j>u$, and for $1\le j\le u-1$ we define r_j inductively by $r_j=\min(\sum_{l=1}^{u-1}i_l-\sum_{l=1}^{j-1}r_l-1,a_j-1)$. In particular, $\phi(m_2)=r_1\ge\min(i_1+i_2-1,a_1-1)=\min(i_1+a_2-2,a_1-1)\ge\min(i_1,a_1-1)\ge i_1=\phi(m_1)$. $\square$

If $C\subseteq S_d$ is a strongly stable set of monomials, then $\Delta C^{(i)}\subseteq C^{(i+1)}$ for $0\le i\le a_1-2$. The next lemma gives a necessary and sufficient condition to have $\Delta C^{(i)}=C^{(i+1)}$ when C is an initial rev-lex segment.

Lemma 3.2. *Let $R\subseteq S_d$ be an initial rev-lex segment and m be the least monomial in R. The following are equivalent:*
(1) $\phi(m)\le r$;
(2) $\Delta R^{(i)}=R^{(i+1)}$ for $r\le i\le a_1-2$.

Proof. First, we will prove the implication (1)$\Rightarrow$(2). Let $s\ge r$. It follows by Lemma 3.1 that the least monomial m' in $\bigcup_{j=s}^{a_1-1}x_1^jR^{(j)}$ has $\phi(m')=s$. Then the least monomial in $\bigcup_{j=s}^{a_1-1}x_1^{j-s}R^{(j)}$ is m'/x_1^s with $\phi(m'/x_1^s)=0$. Moreover, $\bigcup_{j=s}^{a_1-1}x_1^{j-s}R^{(j)}$ is an initial rev-lex segment, which shows that it will be enough to prove only that $\Delta R^{(0)}=R^{(1)}$ in the case $r=0$. Since R is strongly stable, we have that $\Delta R^{(0)}\subseteq R^{(1)}$, so it remains to prove that $\Delta R^{(0)}\supseteq R^{(1)}$. Let $m_1=x_2^{i_2}\cdots x_n^{i_n}\in R^{(1)}$, so $x_1m_1\in R$. Since $x_1m_1>m$ and $m\in R^{(0)}$, it follows that there exists at least one j such that $i_j\le a_j-2$. Let u be the least such j. Then the element $m_2=m_1x_u$ is the largest monomial smaller than x_1m_1 in S_d which is not divisible by x_1, so $m_2\in R$. Since $m_1\in\Delta\{m_2\}$, it follows that $R^{(1)}\subseteq\Delta R^{(0)}$.

Now we will prove the implication (2)$\Rightarrow$(1). Suppose that (1) is not satisfied, so $m=x_1^sm_1$, where $s>r$ and $m_1\in R^{(s)}$. Since by assumption $R^{(s)}=\Delta R^{(s-1)}$, it follows that there exists $m_2\in R^{(s-1)}$, such that $m_2=x_im_1$ for some $i\ge 2$. Then $R\ni x_1^{s-1}m_2=x_1^{s-1}x_im_1<x_1^sm_1=m$, which contradicts the fact that m is the least element in R. $\square$

Note that the conclusion of Lemma 3.2 is not true for arbitrary strongly stable sets. Take for example C to be the smallest strongly stable subset of S_4 containing $x_1x_2x_3^2$ and $x_2^3x_4$. The least element of C is $x_2^3x_4$ with $\phi(x_2^3x_4)=0$. However, $x_2x_3^2\in C^{(1)}\backslash\Delta C^{(0)}$, so $\Delta C^{(0)}\subsetneq C^{(1)}$.

Proof of Theorem 2.2. We give a proof by induction on the number of variables. When $n=1$ the theorem is obvious. Now assume that the theorem is true for $n-1$ variables.

First, we will prove that $|C^{(0)}|\le|R^{(0)}|$. Assume that on the contrary $|C^{(0)}|>|R^{(0)}|$. If p is the least element of S_d, then $p=x_1^{\alpha_1}\cdots x_n^{\alpha_n}$, where $\alpha_n=\min(d,a_n-1)$ and for $1\le i\le n-1$, $\alpha_i=\min(d-\sum_{j=i+1}^n\alpha_j,a_i-1)$. This shows that $\phi(m)\ge\phi(p)$ for any $m\in S_d$, so by Lemma 3.1 it follows that there exists an initial rev-lex segment $\tilde{R}\subseteq S_d$

such that $\tilde{R} \supseteq R$, the least element q in $\tilde{R}$ has $\phi(q)=0$, and $|\tilde{R}^{(0)}| \le |R^{(0)}|+1$. Then $|C^{(0)}| \ge |R^{(0)}|+1 \ge |\tilde{R}^{(0)}|$. By Lemma 3.2 we have that $\Delta\tilde{R}^{(i)} = \tilde{R}^{(i+1)}$ for $0 \le i \le a_1-2$. Since $C^{(0)}$ is a strongly stable set of monomials in $k[x_2,\dots,x_n]_d$ and $R^{(0)}$ is an initial rev-lex segment in $k[x_2,\dots,x_n]_d$ we can apply the induction hypothesis and conclude that $|\Delta C^{(0)}| \ge |\Delta\tilde{R}^{(0)}|$. As C is strongly stable, it follows that $\Delta C^{(i-1)} \subseteq C^{(i)}$ for $1 \le i \le a_1-1$. Therefore $|C^{(1)}| \ge |\Delta C^{(0)}| \ge |\Delta\tilde{R}^{(0)}| = |\tilde{R}^{(1)}| \ge |R^{(1)}|$. Using the induction hypothesis again for $C^{(1)}$ and $\tilde{R}^{(1)}$ we see that $|C^{(2)}| \ge |\Delta C^{(1)}| \ge |\Delta\tilde{R}^{(1)}| = |\tilde{R}^{(2)}| \ge |R^{(2)}|$. Repeating this argument we get that $|C^{(i)}| \ge |\tilde{R}^{(i)}| \ge |R^{(i)}|$ for $1 \le i \le a_1-1$. Then $|C'| = \sum_{i=1}^{a_1-1} |C^{(i)}| \ge \sum_{i=1}^{a_1-1} |R^{(i)}| = |R'|$. However, $|R'| = |R| - |R^{(0)}| > |C| - |C^{(0)}| = |C'|$, which is a contradiction. This proves that $|C^{(0)}| \le |R^{(0)}|$ (and hence that $|C'| \ge |R'|$).

Next we prove (2). As C is strongly stable, it follows that $\Delta C^{(i-1)} \subseteq C^{(i)}$ for $1 \le i \le a_1-1$. Hence $\Delta C = \bigcup_{i=1}^{a_1-1} x_1^{i-1} C^{(i)} \cup x_1^{a_1-1} \Delta C^{(a_1-1)}$, so $|\Delta C| = |C'| + |\Delta C^{(a_1-1)}|$. Similarly, $|\Delta R| = |R'| + |\Delta R^{(a_1-1)}|$. Since we already know that $|C'| \ge |R'|$, it will be enough to prove that $|\Delta C^{(a_1-1)}| \ge |\Delta R^{(a_1-1)}|$. By the induction hypothesis this will in turn follow if $|C^{(a_1-1)}| \ge |R^{(a_1-1)}|$. Assume on the contrary that $|C^{(a_1-1)}| < |R^{(a_1-1)}|$. Since $|C'| \ge |R'|$ it follows that there exists a $t \ge 1$ such that $|C^{(t)}| > |R^{(t)}|$. Applying Lemmas 3.1 and 3.2 again we see as before that there exists an initial rev-lex segment $\tilde{R} \supseteq R$ with the properties that $|\tilde{R}^{(t)}| \le |R^{(t)}|+1$ and $\Delta\tilde{R}^{(i)} = \tilde{R}^{(i+1)}$ for $t \le i \le a_1-2$. Then $|C^{(t)}| \ge |\tilde{R}^{(t)}|$ and by the induction hypothesis we conclude as in the proof of part (1) that $|C^{(i)}| \ge |\tilde{R}^{(i)}| \ge |R^{(i)}|$ for $r \le i \le a_1-1$. But this contradicts our assumption that $|C^{(a_1-1)}| < |R^{(a_1-1)}|$, so $|C^{(a_1-1)}| \ge |R^{(a_1-1)}|$, which proves (2).

Finally, we prove (3). We have that $|C'| + |\Delta C^{(a_1-1)}| = |\Delta C| = |\Delta R| = |R'| + |\Delta R^{(a_1-1)}|$. Since $|C'| \ge |R'|$ and $|\Delta C^{(a_1-1)}| \ge |\Delta R^{(a_1-1)}|$, it follows that $|C'| = |R'|$. Thus $|C^{(0)}| = |C| - |C'| = |R| - |R'| = |R^{(0)}|$, which proves (3). □

Proof of Theorem 2.1. Let $C, R \subseteq S_d$ be the unique sets of monomials such that the image of C (resp. R) in S_d/V (resp. S_d/L) forms a basis of S_d/V (resp. S_d/L). It is easily seen that if we reverse the order of the variables, $x_n < x_{n-1} < \cdots < x_1$, then C becomes strongly stable and R becomes an initial rev-lex segment. Therefore (1) follows from Theorem 2.2 (1).

Since (2) and (3) are easily seen to be true when $n=1$, we can use induction to prove them. So let $n>1$ and assume we have already proved (2) and (3) for $n-1$. Write $V = \tilde{V} \oplus x_n V^{(1)} \oplus \cdots \oplus x_n^{a_n-1} V^{(a_n-1)}$ and $L = \tilde{L} \oplus x_n L^{(1)} \oplus \cdots \oplus x_n^{a_n-1} L^{(a_n-1)}$, where $V^{(i)}, L^{(i)} \subseteq \tilde{S}_{d-i}$ for $1 \le i \le a_n-1$. Then $\dim V^{(i)} + |C^{(i)}| = \dim L^{(i)} + |R^{(i)}| = \dim \tilde{S}_{d-i}$ for all i. The argument in the proof of Theorem 2.2 implies that either $|C^{(i)}| = |R^{(i)}|$ for all i or there exists $1 \le r \le a_n-1$ such that $|C^{(i)}| \le |R^{(i)}|$ for $i \le r-1$, $|C^{(r)}| > |R^{(r)}|$, and $|C^{(j)}| \ge |R^{(j)}|$ for $r+1 \le j \le a_n-1$. Then $\dim V^{(i)} \ge \dim L^{(i)}$ for $i \le r-1$, $\dim V^{(r)} < \dim L^{(r)}$, and $\dim V^{(j)} \le \dim L^{(j)}$ for $r+1 \le j \le a_n-1$. Therefore $\dim x_n V = \dim(x_n \tilde{V} \oplus x_n^2 V^{(1)} \oplus \cdots \oplus x_n^{a_n-1} V^{(a_n-2)}) = \dim V - \dim V^{(a_n-1)} \ge \dim L - \dim L^{(a_n-1)} = \dim x_n L$. We also have by (1) that $\dim \tilde{V} \ge \dim \tilde{L}$, so by the induction hypothesis $\dim \tilde{V}\tilde{S}_1 \ge \dim \tilde{L}\tilde{S}_1$. Therefore $\dim VS_1 = \dim \tilde{V}\tilde{S}_1 + \dim x_n V \ge \dim \tilde{L}\tilde{S}_1 + \dim x_n L = \dim LS_1$, which proves (2).

It remains to prove (3). We will consider two cases: $a_n < \infty$ and $a_n = \infty$. Assume first that $a_n < \infty$. In this case $a_n - 1 \le u$. We have that $VS_1 = \tilde{V}\tilde{S}_1 \oplus x_n V$, $LS_1 = \tilde{L}\tilde{S}_1 \oplus x_n L$, $\dim VS_1 = \dim LS_1$, $\dim \tilde{V}\tilde{S}_1 \ge \dim \tilde{L}\tilde{S}_1$, and $\dim x_n V = \dim V - \dim V^{(a_n-1)} \ge \dim L - \dim L^{(a_n-1)} = \dim x_n L$. Hence $\dim \tilde{V}\tilde{S}_1 = \dim \tilde{L}\tilde{S}_1$ and $\dim V^{(a_n-1)} = \dim L^{(a_n-1)}$. The last equality also implies that $r \le a_n - 2$, so $\dim V^{(a_n-2)} \le \dim L^{(a_n-2)}$. Similarly, we have that $VS_2 = \tilde{V}\tilde{S}_2 \oplus x_n VS_1$, $LS_2 = \tilde{L}\tilde{S}_2 \oplus x_n LS_1$, $\dim VS_2 = \dim LS_2$, $\dim \tilde{V}\tilde{S}_2 \ge \dim \tilde{L}\tilde{S}_2$, and $\dim x_n VS_1 = \dim VS_1 - \dim V^{(a_n-2)} \ge \dim LS_1 - \dim L^{(a_n-2)} = \dim x_n LS_1$. Hence $\dim \tilde{V}\tilde{S}_2 = \dim \tilde{L}\tilde{S}_2$ and $\dim V^{(a_n-2)} = \dim L^{(a_n-2)}$. The last equality implies that $r \le a_n - 3$, so $\dim V^{(a_n-3)} \le \dim L^{(a_n-3)}$. Continuing in this way we obtain that $\dim \tilde{V}\tilde{S}_j = \dim \tilde{L}\tilde{S}_j$ and $\dim V^{(j)} = \dim L^{(j)}$ for $1 \le j \le a_n - 1$. By the induction hypothesis we conclude that $\dim \tilde{V}\tilde{S}_j = \dim \tilde{L}\tilde{S}_j$ for all j. For $j \ge a_n$ we have that $VS_j = \bigoplus_{i=0}^{a_n-1} x_n^i \tilde{V}\tilde{S}_{j-i}$ and $LS_j = \bigoplus_{i=0}^{a_n-1} x_n^i \tilde{L}\tilde{S}_{j-i}$, so $\dim VS_j = \sum_{i=0}^{a_n-1} \dim \tilde{V}\tilde{S}_{j-i} = \sum_{i=0}^{a_n-1} \dim \tilde{L}\tilde{S}_{j-i} = \dim LS_j$.

Now we will prove (3) when $a_n = \infty$. In this case x_n is a nonzerodivisor on S, so $\dim VS_1 = \dim \tilde{V}\tilde{S}_1 + \dim V$ and $\dim LS_1 = \dim \tilde{L}\tilde{S}_1 + \dim L$. Hence $\dim \tilde{V}\tilde{S}_1 = \dim \tilde{L}\tilde{S}_1$. Similarly we conclude that $\dim \tilde{V}\tilde{S}_i = \dim \tilde{L}\tilde{S}_i$ for $1 \le i \le u$. By the induction hypothesis it follows that $\dim \tilde{V}\tilde{S}_i = \dim \tilde{L}\tilde{S}_i$ for all i. Since $\dim VS_j = \dim LS_j$ for $1 \le j \le u$ we can also use induction on j. Fix $j > u$ and assume we have already proved that $\dim VS_{j-1} = \dim LS_{j-1}$. Since $VS_j = \tilde{V}\tilde{S}_j \oplus x_n VS_{j-1}$ and $LS_j = \tilde{L}\tilde{S}_j \oplus x_n LS_{j-1}$, it follows that $\dim VS_j = \dim LS_j$. $\square$

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